readily believe the assertion—that the present amount of the yearly rental of Lancashire would have bought the fee simple of the county in the time of Elizabeth.

It is undoubted that this increase could never have taken place if the total prohibition of usury had continued to exist.

(END OF PART II.)

On the Calculus of Finite Differences, and its Application to Problems in the Doctrine of Compound Interest and Certain Annuities. By WM. CURTIS OTTER, F.R.A.S.

THE calculus of finite differences was created by Taylor, in his celebrated work entitled *Methodus Incrementorum*, and it consists, essentially, in the consideration of the finite increments which functions receive as a consequence of analogous increments on the part of the corresponding variables. These increments or differences, which take the characteristic Δ to distinguish them from differentials, or infinitely small increments, may be in their turn regarded as new functions, and become the subject of a second similar consideration, and so on; from which results the notion of differences of various successive orders, analogous at least in appearance to the consecutive orders of differentials. Such a calculus evidently presents, like the calculus of indirect functions, two general classes of questions:—

1. *To determine the successive differences of all the various analytical functions of one or more variables, as the result of a definite manner of increase of the independent variables, which are generally supposed to augment in arithmetical progression.*

2. *Reciprocally to start from these differences, or, more generally, from any equation established between them, and go back to the primitive functions themselves or to their corresponding relations.*

Hence follows the decomposition of this powerful calculus into two distinct ones, to which are usually given the names of the *direct* and *inverse* calculus of finite differences, the latter being also sometimes called the integral calculus of finite differences.

The differences of this calculus are, by their nature, functions essentially similar to those which have produced them, a circumstance which renders them unsuitable to facilitate the establishment of equations, and prevents their leading to more general relations. Every equation of finite differences is truly, at bottom, an equation

directly relating to the very magnitudes whose successive states are compared. The introduction of new signs, which produce an illusion respecting the true character of these equations, disguises it in a very imperfect manner, since it could always be easily made apparent by replacing the differences by the equivalent combinations of the primitive magnitudes, of which they are really only the abridged notations. I am therefore inclined to think, that the calculus of finite differences is in general improperly classed with the transcendental analysis proper, or calculus of indirect functions. I consider it, on the contrary, in accordance with the views of Lagrange, to be only a very extensive and important branch of ordinary analysis, viz., that of the calculus of direct functions, as the equations to which it applies are always, notwithstanding the notation adopted, simple direct equations.

To sum up, as briefly as possible, the preceding explanations, I consider the calculus of finite differences, when reduced to its most simple general expression, nothing but a complete logical study of questions relating to series. Every series or succession of numbers deduced from one another, in accordance with any constant law, necessarily gives rise to these two fundamental questions:—

1. *The law of the series being supposed known, to find the expression for its general term, so as to be able to calculate immediately any term whatever, without being obliged to form successively all the preceding terms.*

2. *From the same data, to determine the sum of any number of terms of the series by means of their places, so that it may be known without the necessity of continually adding these terms together.*

These two fundamental questions being considered to be resolved, it may be proposed reciprocally to find the law of a series from its general term, or the expression for its sum. Each of these different problems has so much the more extent and difficulty as there can be conceived a greater number of different laws for the series, according to the number of preceding terms on which each term directly depends, and according to the function which expresses that dependence. We may even consider series with several variable indices, as Laplace has done in his *Analytical Theory of Probabilities,* by the analysis to which he has given the name of the "Theory of Generating Functions," which is, really, only a new and more general branch of the calculus of finite differences, or of the general theory of series.

It is now easy to conceive the necessary and perfect identity

between the calculus of finite differences and the theory of series, considered in all its bearings. In fact, every differentiation, after the manner of Taylor, evidently amounts to finding the law of formation of a series with one or with several variable indices from the expression of its general term, in the same way, every analogous integration may be regarded as having for its object the summation of a series, the general term of which would be expressed by the proposed difference; therefore, among the principal general applications which have been made of the calculus of finite differences, it would be but proper to place in the first rank, as the most extended and the most important, the solution of questions relating to series which, independently of its utility as a branch of mathematics, is highly calculated to enlarge the understanding. Series enter more or less into all our branches of analysis, as well as into many of the higher departments of physical science; it is a subject, therefore, not only of curious speculation, but also of the greatest importance, in the various branches of mathematics and philosophy, in consequence of which it has obtained a very considerable share of attention from the most celebrated mathematicians.

One of the most important and useful class of problems in connexion with series are those that occur in the doctrine of interest and annuities, to the solution of which the calculus of finite differences is peculiarly adapted, as the following examples will illustrate. A comparison of the solutions here given with those by the ordinary analysis, affords a striking instance of the truth of De Morgan's remark, on page 11 of the preface to his *Essay on Probabilities*, "that the labour of a person of competent knowledge is seldom lost."

Examples.

1.* To find the amount S_x to which £P will accumulate in x years at compound yearly interest i per £.

Here S_x being the amount at end of the xth year, we have the

* The immense increase of money accumulating at compound interest for a long period, is sufficient to astonish the human mind, and to stagger the credibility of persons who may not be conversant with the properties of geometrical progression, *ex. gr.*:— The amount of a farthing placed out at compound interest at the commencement of the Christian era, and continued to the end of the eighteenth century, would be 144,035 quintillions of pounds; but of the magnitude of this sum, spoken of in the abstract, no just conception can be formed. When, however, by a further calculation we ascertain, that to coin such a quantity of money (were it possible) into sovereigns of the present weight and fineness, we should require 60,308,170 solid globes of gold, each as large as the earth, we are enabled to entertain a more adequate idea of the sum, whose vastness, without having recourse to this adscititious assistance, placed it almost beyond the reach of our limited understandings.

following equation expressing the relation between the amounts at the end of the x and $(x+1)$th year:—

$$S_{x+1}=S_x+i\,S_x;$$
$$\text{or, } S_{x+1}-S_x(1+i)=o.$$

$\therefore$ Integrating, we get $S_x=C.(1+i)^x$,

$$\therefore\ S_{x=o}=P=C,$$
$$\therefore\ S_x=P.(1+i)^x,$$

the amount sought, which is the same as that obtained by the ordinary method.

2. To find the present value of £P due at the end of x years, at i per £ compound yearly interest.

Let v_{x+1}=the present value at end of $\overline{x+1}$ years, then we have equation of condition,

$$v_{x+1}+i\,v_{x+1}=v_x;$$
$$\text{or, } v_{x+1}-v_x(1+i)^{-1}=o.$$
$$\therefore\ v_x=C(1+i)^{-1}.$$

When $x=o$, $C=P$,

$$\therefore\ v_x=P.(1+i)^{-1},$$

being the ordinary rule.

Obs.—I may mention, that the advantage of this calculus is not so forcibly illustrated in the solution of such elementary problems as those just discussed, as in those of a more difficult class, as will be hereafter seen. Indeed, the great value of this important calculus consists in the fact, that it is equally adapted to the solution of the most elementary problems, and to that of the most complicated and abstruse.

3. To find the amount of a yearly annuity of £μ, payable for x years at compound yearly interest of i per £.

Let A_x=the amount at end of x years, then the following relation exists between the amount at the end of the xth and $(x+1)$th year:—

$$A_{x+1}=A_x+i\,A_x+\mu;$$
$$\text{or, } A_{x+1}-A_x(1+i)-\mu=o.$$

Integrating, we have

$$A_x=C(1+i)^x-\frac{\mu}{i}$$
$$\therefore\ A_{x=o}=o=C-\frac{\mu}{i}\ \therefore\ C=\frac{\mu}{i},$$
$$\therefore\ A_x=\frac{\mu}{i}(1+i)^x-\frac{\mu}{i}=\mu\frac{(1+i)^x-1}{i},$$

which is the ordinary rule.

4. To find the present value of a yearly annuity of £μ for x years, at compound yearly interest of i per £.

Let v_x = the present value at the end of x years, then we have

$$v_x = v_{x+1} + iv_{x+1} - \mu;$$

$$\text{or, } v_{x+1} - v_x(1+i)^{-1} - \mu(1+i)^{-1} = o.$$

$\therefore$ Integrating, we get

$$v_x = C(1+i)^{-x} + \frac{\mu}{i},$$

$$\therefore v_o = o = C + \frac{\mu}{i} \therefore C = -\frac{\mu}{i},$$

$$\therefore v_x = \mu . \frac{1-(1+i)^{-x}}{i},$$

being the ordinary rule.

Obs.—In some cases, the solution may be more easily effected by integrating the general term of the series answering to the conditions of the problem given for solution, the principle of which may be demonstrated as follows:—

Let S_x represent the sum of the first x terms of a series whose general term $= u_x$, then

$$S_x = u_1 + u_2 + u_3 + u_4 + \ldots u_x,$$

$$\text{and } S_{x+1} = u_1 + u_2 + u_3 + u_4 + \ldots u_x + u_{x+1};$$

$$\therefore S_{x+1} - S_x = \Delta S_x = u_{x+1},$$

$$\therefore S_x = \Sigma u_{x+1} + C.$$

When $x = o$, $S_o = o = \Sigma u_1 + C \therefore C = -\Sigma u_1$,

$$\therefore S_x = \Sigma u_{x+1} - \Sigma u_1.$$

Applying this method to the solution of the two last problems, we have—

$$u_x = \mu(1+i)^{x-1} \quad . \quad . \quad . \quad (1)$$

$$\text{and } u_x = \mu(1+i)^{-x} \quad . \quad . \quad . \quad (2)$$

Since the amount, or present value, of any annuity for any given number of years equals the sum of the amounts or present values of the instalments of the annuity as they become due,

Hence, in the first case, we have from equation . . (1)

$$u_{x+1} = \mu(1+i)^{x+1},$$

$$\text{and } S_x = \Sigma u_{x+1} + C = \mu . \frac{(1+i)^x}{i} + C;$$

$$\therefore S_o = o = \frac{\mu}{i} + C \therefore C = -\frac{\mu}{i}.$$

Hence,

$$S_x = \mu . \frac{(1+i)^x - 1}{i},$$

the same as obtained from the integration of the equation of difference answering to the conditions of this question.

In the second case, we have from equation . . . (2)

$$u_{x+1}=\mu.(1+i)^{-x-1}$$

$$\therefore\ S_x=\Sigma u_{x+1}+C=\mu.\frac{(1+i)^{-x-1}}{(1+i)^{-1}-1}+C$$

$$=\mu\frac{(1+i)^{-x}}{-i}+C,\text{ when } x=o,\ C=\frac{\mu}{i},$$

$$\therefore\ S_x=\mu.\frac{1-(1+i)^{-x}}{i},$$

being the same as before obtained from the integration of the equation of difference answering to this question.

5. Find the amount of an increasing annuity for x years, commencing with £μ and increasing £h every year, at i per £ compound interest.

Let S_x be the amount at the end of the xth year, then we have the following equation of condition:—

$$S_{x+1}=S_x+iS_x+(\mu+hx);$$

$$\text{or, } S_{x+1}-S_x(1+i)-(\mu+hx)=o.$$

Integrating, we get

$$S_x=(1+i)^{x-1}.\left\{\Sigma\frac{\mu+hx}{(1+i)^x}+C\right\}$$

$$=(1+i)^{x-1}.\left\{-\frac{\mu+xh}{i(1+i)^{x-1}}-\frac{h}{i^2(1+i)^{x-1}}+C\right\}$$

$$=-\frac{\mu+xh}{i}-\frac{h}{i^2}+C(1+i)^{x-1};$$

when $x=o$, $C=(1+i).\left(\frac{h+i\mu}{i^2}\right)$,

$$\therefore\ S_x=\frac{(1+i)^x.(h+i\mu)-(h+i\mu+ihx)}{i^2}=$$

$$=\frac{h+i\mu}{i}\left\{\frac{(1+i)^x-1}{i}\right\}-\frac{hx}{i}$$

$$=\frac{A_x(h+i\mu)-hx}{i}\quad . \quad . \quad . \quad . \quad (1)$$

Where A_x= the amount of an annuity of £1 per annum for x years at the given rate of interest.

Cor. 1.—By simply changing the sign of h in equation (1), we get the value of a *decreasing* annuity commencing at £μ, and decreasing £h every year, viz.:—

$$S_x=\frac{A_x(i\mu-h)+hx}{i}\quad . \quad . \quad . \quad . \quad (2)$$

Cor. 2.—When $\mu=h$, we have from equation (1),

$$S_x=\frac{h}{i}\left\{A_x(1+i)-x\right\} \quad . \quad . \quad . \quad . \quad (3)$$

and from equation (2),

$$S_x=\frac{h}{i}\left\{A_x(i-1)+x\right\} \quad . \quad . \quad . \quad . \quad (4)$$

6. Find the present value of an increasing annuity for x years, commencing at £μ, and increasing £h every year, at i per £ compound interest.

The general term of this annuity is evidently $=\dfrac{\mu+(x-1)h}{(1+i)^x}$;

hence, $$\mu_{x+1}=(\mu+xh)(1+i)^{-x-1}$$

$$\therefore S_x=\Sigma\mu_{x+1}+C=\frac{(\mu+xh)(1+i)^{-x-1}}{-i(1+i)^{-1}}-\frac{h.(1+i)^{-x-2}}{i^2(1+i)_2}+C$$

$$=\frac{(\mu+xh)(1+i)^{-x}}{-i}-\frac{h.(1+i)^{-x}}{i^2}+C;$$

when $x=o$, $C=\dfrac{h+i\mu}{i^2}$,

$$\therefore S_x=\frac{h+i\mu}{i^2}-\frac{h(1+i)^{-x}}{i^2}-\frac{(\mu+xh)(1+i)^{-x}}{i}$$

$$=\frac{h+i\mu}{i^2}-\frac{h(1+i)^{-x}}{i^2}-\frac{i\mu(1+i)^{-x}}{i^2}-\frac{xh}{i}(1+i)^{-x}$$

$$=\left(\frac{h+i\mu}{i}\right).\left(\frac{1-(1+i)^x}{i}\right)-\frac{xh}{i}(1+i)^{-x}$$

$$=V_x.\left(\frac{h+i\mu}{i}\right)-\frac{xh}{i}R^{-x} \quad . \quad . \quad . \quad . \quad (1)$$

Where V_x = the present value of £1 per annum for x years, and R^{-x} the present value of £1 due at the end of x years.

Cor. 1.—By changing the sign of h, in formula (1), we have an expression for the value of the corresponding *decreasing* annuity, viz.:

$$S_x=V_x\left(\frac{i\mu-h}{i}\right)+\frac{xh}{i}R^{-x}. \quad . \quad . \quad . \quad . \quad (2)$$

Cor. 2.—If $h=\mu$, we have from equation (1),

$$S_x=\frac{h}{i}\left\{V_x(1+i)-xR^{-x}\right\}. \quad . \quad . \quad . \quad . \quad (3)$$

and from equation (2),

$$S_x=\frac{h}{i}\left\{V_x(i-1)+xR^{-x}\right\}. \quad . \quad . \quad . \quad . \quad (4)$$

7. A person invests a sum of money (S) at i per £ compound interest, and expends yearly a certain portion of the interest (a), adding the remainder to the stock, what is the amount at the end of x years?

Let A_x represent the amount sought, then, per question, we have the following equation of condition:—

$$A_{x+1}=A_x+A_xi-a;$$
$$\text{or, } A_{x+1}-A_x(1+i)+a=o.$$

∴ Integrating, we get

$$A_x=C(1+i)^x+\frac{a}{i}.$$

Now, when $x=o$, $C=S-\frac{a}{i}$,

$$\therefore A_x=\left(S-\frac{a}{i}\right)(1+i)^x+\frac{a}{i}$$
$$=S(1+i)^x-a\frac{(1+i)^x-1}{i},$$

being the accumulation of the capital S in the time x, less the amount of an annuity equal to his annual expenditure, which is obviously correct.

8. A person spends every year twice the sum he gained in trade the previous year; his business, however, becomes every year more profitable, and he finds his property increase regularly as the square of the time from the commencement of business: in what ratio do the profits of his trade increase?

Assuming P_x to be the profit of the xth year, the problem leads to the following equation:—

$$P_{x+1}-2P_x=a(2x+1).$$

∴ Integrating, we get

$$P_x=a\left\{5.2^{x-1}-(2x+3)\right\}+P_1.2^{x-1}.$$

Now, $P_1=a.1^2$,

$$\therefore P_x=a\left\{3.2^x-2x-3\right\},$$

which gives the ratio sought.

Obs.—In the next Number of the *Magazine*, I hope to discuss some questions depending on the theory of circulating functions. In all the preceding equations the coefficients are continuous functions, but such continuity is not necessary in the way they have been used. *Ex. gr.:* in the general equation $u_{x+2}-xu_{x+1}+x^2u_x-x^3=o$, it is clearly unnecessary that the function of x should be of the same form when x is a fraction; for the equation, its solution, and process of verification, are all independent of such values. Neither is it necessary that the coefficients should retain the same form when x is an integer, for it may be shown that results can be deduced of a finite form, when they circulate through any number of different forms, as x varies in value.